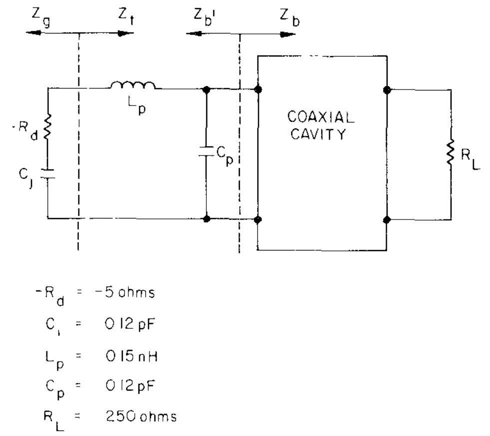

Fig. 5. Equivalent circuit for oscillator.

IV. Analysis of Results

Referring to Test 5 with image guide as in Fig. 2(a), detailed measurements were made of power and frequency of the oscillator as a function of coaxial cavity height d. Here, d is the distance from the bottom of the cavity to the bottom of the iris. To convert this length to l, the distance from the bottom of the cavity to the top tuning disk, add 0.55 in as in Fig. 2(a). The test results are shown in Fig. 4. These data are taken with no external tuning except changing the coaxial cavity height. Note as the coaxial cavity is varied from the maximum height to the minimum height, the frequency variation is nearly linear until a maximum frequency is reached. Then the cycle starts again and the frequency is repeated every half wavelength.

A similar test was tried on the same mechanical structure except that, here, external tuning using pieces of metal placed near the dielectric were used in addition to varying the cavity height (Test 5). Hence, the power output is considerably higher, but the frequency shows the same periodicity.

In Test 2, a microstrip line was used with an alumina top housing. Additional tuning with external elements was used. Note the wide tuning range (11 GHz) and the power levels up to 11 mW obtained and, again, a similar periodicity. In Test 3, microstrip line was used with a metal cylinder top housing. Note that the tuning range was diminished to less than 1 GHz and the power level remained nearly the same, i.e., 7 mW.

V. Circuit Analysis

The equivalent circuit for these oscillators is shown in Fig. 5. Here, $-R$ represent the negative resistance of the Gunn diode, C_j is the junction capacitance of the diode, and L_p and C_p are the package inductance and capacitance parasitics. The coaxial cavity is represented by a two-port network, consisting of a transmission line for the coaxial cavity and R_L is the output load resistance. The load resistance is assumed to be high, since the output impedance is matched to air or to a metal waveguide for measurement. The R_L is assumed to be transformed to the terminals across C_p. Another reason for R_L being high is that for oscillation to occur, the $\mathrm{Re}(Z_t)$ must be less than $-R$, which is about 5 Ω. The details of this analysis, where the criterion for oscillation requires high R_L were shown in previous reports [1]–[3].

VI. Conclusions

An oscillator has been developed useful for launching electromagnetic energy into image guide or microstrip transmission line. This is done by recessing the Gunn diode deeply in the ground plane in order to establish a coaxial resonating device. The open guided structure was then placed over the coaxial opening in order to guide the energy out of the oscillator structure. Oscillators were designed with up to 43 mW at 57 GHz, with diodes which provided, in metal waveguide oscillators, up to 70-mW power output. In addition, mechanical tuning provided a linear range of over 11 GHz with a continuously variable height cavity. Results are very reproducible from one oscillator to another without external compensation.

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A Simple Numerical Method for Studying the Propagation Characteristics of Single-Mode Graded-Index Planar Optical Waveguides

A. N. KAUL, S. I. HOSAIN, AND K. THYAGARAJAN

***Abstract*—A simple numerical method based on the Runge–Kutta method is presented to compute the propagation constant, the modal field, and the cutoff wavelength corresponding to the fundamental TE_0 and TM_0 modes of a planar optical waveguide with an arbitrary refractive index profile. The method is much simpler and requires less computational effort than the earlier reported numerical methods. We have also used the technique for an estimation of the effect of the $\nabla\epsilon$ term in TM modes.**

I. Introduction

In recent years, a considerable amount of work has been reported on the study of the propagation characteristics of inhomogeneous planar optical waveguides used in integrated optics [1]–[12]. Most of the waveguides used in integrated optics are single moded [16], which support the fundamental TE_0 and TM_0 modes. In order to optimize the performance of integrated optical devices using such waveguides, it is important to know the propagation characteristics of such waveguides.

The propagation constant and the transverse modal field distribution of a mode can, in general, be obtained from the solution

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of the Maxwell's equations. However, except for a few refractive index profiles (viz., exponential, secant–hyperbolic, parabolic, etc.), analytical solutions do not exist for such waveguides. Hence, one uses either approximate methods such as the perturbation method [2], the WKB method [3]–[5], and the variational method [6], or uses numerical methods [7]–[14] to obtain the solution of this equation. In many cases, the perturbation method cannot be applied successfully, as there is no closely related problem for which an exact solution exists. The WKB method does not give accurate results for lower order modes [13] or for steep gradient refractive index profiles. For the variational method, the choice of the trial field is very critical and it should resemble the exact field very closely. In addition, since the variational and WKB methods become inaccurate close to cutoff, the cutoff value determination requires more accurate techniques. The numerical methods reported so far, although quite useful, seem to be complicated, require extensive computations, and do not give any satisfactory criterion for achieving the required accuracy. For instance, in the method proposed by Pichot [8], the Fredholm equation approach is used to evaluate $n_e(=\beta/k_0)$, which is quite cumbersome and there is no discussion about the computing time with respect to accuracy. In the method proposed by Ramaswamy *et al.* [11], calculations have been done for multimode waveguides only; however, in actual practice, waveguides used in integrated optical circuits are single moded. In addition, the WKB technique has been used to determine n_e, which becomes inaccurate for low V values [3], [4] and for profiles far from parabolic [9]. As mentioned by them, their method fails for large depths due to the blowing up of the fields. Also, the cutoff calculations have been done using the WKB approach, which again is not very accurate. In the method proposed by Meunier *et al.* [12], calculations have been done only for multimode waveguides and there is no mention about the computing time with respect to accuracy, etc.

In this paper, we present a simple numerical method for the solution of the wave equation corresponding to the TE_0 and TM_0 modes of an inhomogeneous planar optical waveguide with arbitrary refractive index profile. The method is extended to compute the cutoff wavelength of the TE_1 mode which determines the limit of single-mode operation. Such numerical methods are very useful to test the validity of the commonly used approximate methods [2]–[6]. In Section II, we will briefly discuss the necessary theory of our method, and in Section III, we will present some numerical results to show the validity of our method.

II. Theory

We consider an inhomogeneous planar optical waveguide for which the refractive index distribution can be written as

$$n^2(x) = n_s^2 + 2n_s\Delta f(x), \qquad x \geqslant 0$$
$$= n_c^2, \qquad x < 0 \tag{1}$$

where n_s is the refractive index of the substrate, n_c is the refractive index of the cover (which is usually air), $n_s + \Delta$ is the maximum refractive index (for $\Delta \ll n_s$), and $f(x)$ defines the shape of the profile; $f(o) = 1$ for all profiles and, except for parabolic and linear profiles, $f(\infty) = 0$. For an exponential profile $f(x) = \exp(-x/d)$, and for a Gaussian profile $f(x) = \exp(-x^2/d^2)$, where d is the diffusion depth. For the linear profile $f(x) = 1 - x/d$ and for the parabolic profile $f(x) = 1 - x^2/d^2$, so that, for such profiles, $f(d) = 0$ [11].

The transverse field component ϕ of the mode satisfies the following eigenvalue equation [17]:

$$\frac{d^2\phi}{d\xi^2} + k_0^2 d^2\left[n^2(\xi) - n_e^2\right]\phi = 0 \qquad \text{for TE mode} \tag{2a}$$

$$\frac{d^2\phi}{d\xi^2} + \left[k_0^2 d^2 n^2(\xi) - \frac{3}{4}\left(\frac{1}{n^2}\frac{dn^2}{d\xi}\right)^2 + \frac{1}{2n^2}\frac{d^2n^2}{d\xi^2} - \beta^2 d^2\right]\phi = 0 \qquad \text{for TM mode} \tag{2b}$$

where $\phi = E_y$ for TE modes and $\phi = H_y/n(\xi)$ for TM modes, n_e $(=\beta/k_0)$ is the effective index of the mode, β the propagation constant, $\xi = x/d$, k_0 is the free-space wavenumber, E_Y is the transverse electric field of the TE mode, H_Y is the transverse magnetic field of the TM mode, and the direction of propagation is along the Z-axis.

A. Computation of n_e

A numerical method for the solution of the scalar wave equation corresponding to the fundamental mode of radially inhomogeneous optical fibers has been discussed in detail by Sharma *et al.* [18]. Following a similar approach and substituting $n^2(\xi)$ from (1) in (2a) and (2b) for the TE and TM modes, respectively, we transform the second-order differential equations into the following first-order differential equations:

$$\frac{dG}{d\xi} = -G^2 - k_0^2 d^2\left[n_s^2 + 2n_s\Delta f(\xi) - n_e^2\right]\xi, \qquad \xi \geqslant 0 \text{ for TE mode} \tag{3}$$

$$\frac{dG}{d\xi} = -G^2 - k_0^2 d^2\left[n^2(\xi) - \frac{3}{4k_0^2 d^2}\left\{\frac{1}{n^2(\xi)}\frac{dn^2(\xi)}{d\xi}\right\}^2 + \frac{1}{2k_0^2 d^2 n^2(\xi)}\frac{d^2 n^2(\xi)}{d\xi^2} - n_e^2\right], \qquad \xi \geqslant 0 \text{ for TM mode} \tag{4}$$

where $G = 1/\phi \cdot d\phi/d\xi$. In order to obtain the boundary conditions on G, we observe that, for TE modes, both ϕ and $d\phi/d\xi$ are continuous everywhere. Thus, G is continuous everywhere. Since in the cover region ($\xi < 0$)

$$\phi \sim \exp\left[k_0 d\sqrt{n_e^2 - n_c^2}\,\xi\right] \tag{5a}$$

we obtain

$$G(\xi = +0) = G(\xi = -0) = \frac{1}{\phi}\frac{d\phi}{d\xi}\bigg|_{\xi=-0} = k_0 d\sqrt{n_e^2 - n_c^2}. \tag{5b}$$

However, for TM modes, H_y and $\dfrac{1}{n^2(\xi)}\dfrac{dHy}{d\xi}$ are continuous everywhere, since $\phi = H_y/n(\xi)$, we have

$$\phi(\xi = +0) = \frac{H_y(\xi = +0)}{\left(n_s^2 + 2n_s\Delta\right)^{1/2}} = \frac{H_y(\xi = -0)}{\left(n_s^2 + 2n_s\Delta\right)^{1/2}} = \frac{n_c\phi(\xi = -0)}{\left(n_s^2 + 2n_s\Delta\right)^{1/2}}. \tag{5c}$$

Similarly

$$\frac{1}{\phi}\frac{d\phi}{d\xi}\bigg|_{\xi=+0} = \frac{1}{H_y}\frac{dH_y}{d\xi}\bigg|_{\xi=+0} - \frac{1}{2n^2(\xi)}\frac{dn^2(\xi)}{d\xi}\bigg|_{\xi=+0}$$
$$= \left(n_s^2 + 2n_s\Delta\right)\frac{1}{\phi}\frac{d\phi}{d\xi}\bigg|_{\xi=-0} - \frac{n_s\Delta}{\left(n_s^2 + 2n_s\Delta\right)}\frac{df}{d\xi}\bigg|_{\xi=-0}.$$

Thus

$$G(\xi=+0)=\frac{k_0 d}{n_c^2}\left(n_s^2+2n_s\Delta\right)\left(n_e^2-n_c^2\right)^{1/2}-\frac{n_s\Delta}{\left(n_s^2+2n_s\Delta\right)}\left(\frac{df}{d\xi}\right)_{\xi=+0}. \quad (5d)$$

Far into the substrate (i.e., $\xi\to\infty$), since the refractive index saturates to n_s, the field is expected to tend as $\exp[-k_0 d\sqrt{n_e^2-n_s^2}\,\xi]$ and thus

$$G(\xi\to\infty)=-k_0 d\sqrt{n_e^2-n_s^2}. \quad (6)$$

The above boundary condition for $G(\xi\to\infty)$ is valid for profiles which saturate continuously to n_s. On the other hand, for the linear and parabolic profiles, since $n=n_s$ for $x>d$, the boundary condition becomes $G(\xi=1)=-k_0 d\sqrt{n_e^2-n_s^2}$.

Hence, for a given planar waveguide (i.e., for given values of k_0, d, n_s, n_c, Δ, and $f(\xi)$), the problem of computing n_e for a particular mode reduces to solving the first-order differential equation (3) or (4) as the case may be, using any standard procedure starting with the initial condition (5b) or (5d) and satisfying (6) as $\xi\to\infty$. Theoretically, $G(\xi\to\infty)$ corresponds to the value of G at a very large value of ξ; but in order to reduce the number of steps in the computation, we assume the point $\xi\to\infty$ to be some finite value of ξ much greater than unity (i.e., $x\gg d$) where G is very small. This value would depend on the accuracy required in the computation of n_e.

B. Computation of TE_0 and TM_0 Mode Cutoffs

The cutoff frequencies of the TE_0 and TM_0 modes are parameters of considerable interest. At cutoff, $n_e=n_s$; hence, (3) and (4) reduce to

$$\frac{dG}{d\xi}=-G^2-2k_{0c}^2d^2n_s\Delta f(\xi),\qquad \xi>0 \text{ for } TE_0 \text{ mode} \quad (7a)$$

$$=-G^2-k_{0c}^2d^2\left[2n_s\Delta f(\xi)-\frac{3}{4k_{0c}^2d^2}\left\{\frac{1}{n^2(\xi)}\frac{kn^2}{d\xi}\right\}^2+\frac{1}{2k_{0c}^2d^2n^2(\xi)}\frac{d^2n^2}{d\xi^2}\right],\qquad \xi>0 \text{ for } TM_0 \text{ mode} \quad (7b)$$

with the boundary conditions again given by (5) and (6) and n_e replaced by n_s and k_0 replaced by $k_{0c}=2\pi/\lambda_c$, where λ_c is the cutoff wavelength. In addition, $G(\xi\to\infty)=0$ for both TE_0 and TM_0 modes. Having obtained the cutoff wavelength, one can immediately obtain the cutoff V number of the waveguide

$$V_c=k_{0c}d(2n_s\Delta)^{1/2}.$$

The problem of computing λ_c for TE_0 and TM_0 modes then reduces one to solving (7a) or (7b) with the above-mentioned conditions on $G(\xi=+0)$ and $G(\xi\to\infty)$. As before, the point $\xi=\infty$ is chosen as some finite value of ξ much greater than unity.

C. Computation of TE_0 and TM_0 Modal Fields

Since ϕ satisfies a linear differential equation, the solution is unique apart from a constant multiplier. Hence, without loss of generality, we may assume the field to be unity at the cover-guide interface, i.e.,$\phi(0)=1$. The value of ϕ at an arbitrary value of ξ can be written as

$$\phi(\xi)=\exp\left[\int_0^{\xi}G(\xi')\,d\xi'\right]. \quad (8)$$

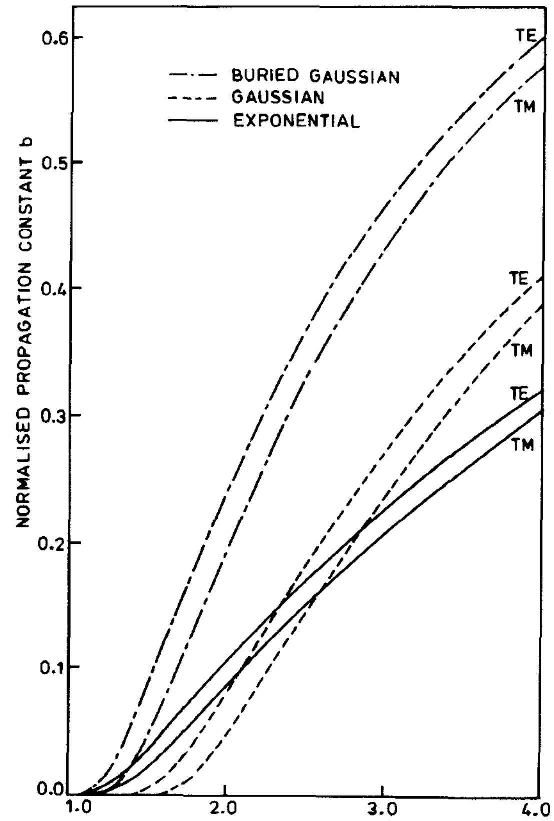

Fig. 1. Plot of normalized propagation constant b as a function of V for TE_0 and TM_0 modes for the three index profiles.

Once n_e is determined, the quantity $\int_0^{\xi}G(\xi')d\xi'$ is computed numerically, where the integrand $G(\xi')$ is obtained from the numerical solution of (3) or (4) corresponding to the TE_0 or TM_0 mode. Thus, the modal field $\phi(\xi)$ can be plotted as a function of ξ.

D. Computation of TE_1 Mode Cutoff

The cutoff frequency of the TE_1 mode determines the upper limit of single-mode operation and, hence, is a parameter of considerable interest. The modal field ϕ for the TE_1 mode cutoff satisfies the same equation (7a) with the same boundary conditions. Thus, the cutoff V value of the TE_1 mode is obtained in a similar manner. However, it may be noted that, for the TE_1 mode, at some value of $\xi<\infty$, ϕ becomes zero, which implies $G\to\infty$. Hence, in this region, one has to switch over to the substitution $F=1/G=\phi(d\phi/d\xi)^{-1}$ which transforms (7a) to the following form:

$$\frac{dF}{d\xi}=1+2F^2R_{0c}^2d^2n_s\Delta f(\xi). \quad (9)$$

Once the $F=0$ ($G\to\infty$) region is crossed, (7a) is used again to compute $G(\infty)$.

III. Numerical Results and Discussion

In this section, we present some numerical results to test the validity of our method. We have computed the effective indices, the modal field, and cutoff V values for the TE_0 and TM_0 modes corresponding to an exponential, a Gaussian, and a buried Gaussian profile for which $f(x)$ is $e^{-x/d}$, e^{-x^2/d^2}, and $e^{-(x-a)^2/d^2}$, respectively. We assume typical values of the various parameters: $n_s=2.177$, $\Delta=0.043$, $d=0.931\mu$m, $a=0.2009\mu$m,

TABLE I
CUTOFF V VALUES OBTAINED BY PRESENT METHOD

Mode	$f(x)=e^{-x/d}$	$f(x)=e^{-x^2/d^2}$	$f(x)=e^{-(x-a)^2/d^2}$
TE_0	1.0868	1.4333	1.0973
TM_0	1.1735	1.5843	1.2224
TE_1	2.6593	4.0376	3.2218

$n_c = 1.0$, and $V = 2.0$, which, as we will show, corresponds to the single-mode region. The value of n_e for the TE_0 mode corresponding to the exponential profile obtained by our method is 2.1815083, while the exact and the WKB values are 2.1815083 and 2.1805704, respectively. Thus, we see that our results match exactly with the exact, while WKB gives an error of about 9×10^{-4}. Similarly, the value of n_e for the TE_0 mode corresponding to the Gaussian profile obtained by our method is 2.1805099, while the WKB value is 2.1774472. Thus, the WKB method gives an error of 3×10^{-3}.

In Fig. 1, the normalized propagation constant

$$b\left(=\frac{n_e^2 - n_s^2}{2n_s\Delta}\right)$$

obtained by the present method has been plotted as a function of V for TE_0 and TM_0 modes corresponding to the above three profiles.

In Table I, we have presented the cutoff V values of the TE_0, TM_0, and TE_1 modes corresponding to the above three profiles. As expected, we observe from Table I that, for each of these profiles, the cutoff V value of the TM_0 mode is greater than that of the TE_0 mode.

We may mention here that the values of N and the maximum value of ξ depend on the accuracy required. We have verified that for the profiles studied here convergence is obtained with $N = 40$ for TE and $N = 60$ points for TM modes per unit interval in ξ. An accuracy of 10^{-5} in n_e requires typically a maximum value of ξ of about 4 and taking a CPU time of about 35 seconds on ICL 2960 computer. On the other hand, an accuracy of 10^{-7} in n_e can be obtained with the same N value but the required CPU time goes up to 55 s. In addition, since the program is self-iterative (i.e., it iterates on the maximum ξ starting from some initial value), there are no blowing up of the field problems as reported in [11].

As can be seen from (2b) for graded refractive index profiles, one has additional terms containing $dn^2/d\xi$ and $d^2n^2/d\xi^2$ which are absent in homogeneous waveguides. Using our technique, we can indeed estimate the effect of these terms and we have found that for the profiles considered the effect of these terms is only about .0014 percent near cutoff and .001 percent far from cutoff. Hence, for such profiles, one may neglect the effect of the $\nabla\epsilon$ term.

In Fig. 2(a) and (b), we have plotted the normalized transverse modal fields E_y and H_y as a function of ξ for the TE_0 and TM_0 modes, respectively, corresponding to the exponential and buried Gaussian profiles for $V = 2.0$. Such plots are extremely useful in computing the field depth which is defined as the value of ξ where the field value reduces to $1/e$ of its peak value. The modal field distributions are also useful in overlap integral calculations for estimating the efficiencies in electrooptic or acoustooptic interactions using such waveguides, coupling efficiency calculations, etc.

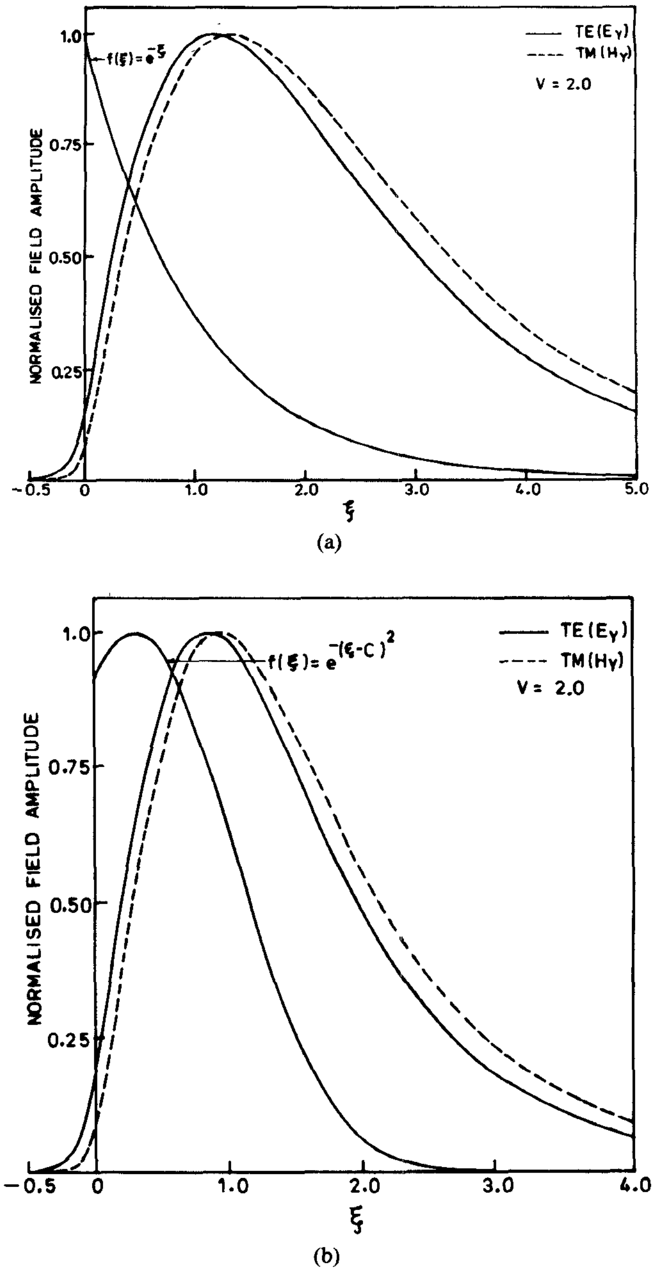

Fig. 2. Plot of normalized field amplitude as a function of $\xi(=x/d)$ for TE_0 and TM_0 modes for (a) exponential profile and (b) buried Gaussian profile with $n_s = 2.177$, $n_c = 1.0$, $\Delta = 0.043$, $d = 0.931\mu m$, $a = 0.2909\mu m$, and $V = 2.0$.

IV. SUMMARY

In this paper, we have presented a simple numerical method to compute the propagation characteristics of single-mode inhomogeneous planar optical waveguides with arbitrary refractive index profiles. The method not only converges rapidly but is also capable of giving results of specified accuracy. The results obtained by the present method are useful in checking the accuracies of the results obtained by various approximate methods. The method can be used in effective index calculations where usually the effective index profile is quite arbitrary and is known only at a finite number of points. Hence, the method can be used with a little modification for such calculations. The method can also be used for any experimentally determined profile.

Exact fields are obtained by the present method which can be used to form trial fields for variational calculations. Besides, the same method can be used for fundamental as well as other higher order modes to obtain n_e and field profiles.

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Calculation of Cutoff Wavenumbers for TE and TM Modes in Tubular Lines with Offset Center Conductor

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Abstract —**The cutoff wavenumbers of TE and TM modes (higher order modes) in a tubular line having an offset center conductor have been calculated. Whereas most previous methods used to study this structure were of an approximate nature, the analytical method developed by Singh and Kothari leads to a rigorous analytical formulation. The boundary conditions on both conductor boundaries, assumed to be perfectly conducting, are satisfied exactly. The cutoff values calculated show that some results previously reported are inaccurate.**

I. Introduction

Introducing a lateral offset in the center conductor of a coaxial line provides a simple way to decrease its characteristic impedance without modifying the dimensions of the conductors [1], [2]. This technique can be used to realize quarter-wave transformers and other matching devices. The properties of the dominant TEM mode can readily be determined with conformal mapping. Furthermore, it is also necessary to determine the cutoff frequency of higher order modes which set an upper limit to the useful frequency of operation.

The propagation along this geometry was considered by several authors [2]–[6] using approximate techniques for its analysis (in particular, point-matching and conformal mapping). While most articles did not indicate which accuracy was obtained, one recent article [6] provides an upper and a lower bound. In some instances, however, the range between the bounds is rather large (up to 20 percent), making the use of the results of little practical interest. For some other situations, the bounds for successive solutions actually overlap one another.

The same problem was tackled analytically by other authors [7]–[9]. A special perturbation method was developed in [8]. It could be useful when extended to dielectric waveguides or eccentric Goubau lines [10], but the study considers only small eccentricities. Some of the tabulated parameters of [8] actually yield nonphysical results; also, symmetric and antisymmetric modes appear to be degenerate, which contradicts experimental observations. Finally, an analytical method devised to analyze the related problem of a circular plate with an eccentric circular hole [9] yields incorrect final expressions. Detailed comments on this paper have appeared recently [17].

The analysis of previous publications shows that, even though considerable effort has been devoted to the study of this geometry, the available techniques are still either approximate when not altogether incorrect.

A rigorous mathematical derivation is presented in the present paper. The Helmholtz equation for higher order modes is solved exactly, and the boundary conditions on the two offset conductors are satisfied by the technique developed by Singh and Kothari [11], based on Graf's addition theorem for Bessel functions [12]. One obtains in this manner an infinite set of linear equations which must be truncated to permit numerical calculations. The accuracy of the results can be arbitrarily improved upon by taking additional terms.

II. Basic Theory

The longitudinal direction of the offset tubular line, to which the axes of the two conductors are parallel, is the z direction. The system is symmetrical with respect to the x axis; its transverse cross section is shown in Fig. 1, in which all significant dimensions are also reported. Two polar coordinate systems, labeled (r, θ) and (r', θ'), are defined with respect to the centers of the two conductors located at 0 and 0′, respectively.

The general solution for the transverse dependence of the potential is obtained by solving the two-dimensional Helmholtz

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